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Relative commutants of finite groups of unitary operators and commuting maps [☆,☆☆]

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For a von Neumann algebra $\mathcal{R}$ we determine the commutant of the set $\{u \otimes u : u \in \mathcal{R},\ u \text{ unitary}\}$ and normal functionals on $\mathcal{R}\overline{\otimes}\mathcal{R}$ that are invariant under all automorphisms implemented by $u \otimes u$ for u unitary in $\mathcal{R}$. For a finite group G of unitary operators on a Hilbert space $\mathcal{H}$ implementing automorphisms of a von Neumann algebra $\mathcal{S} \subseteq \mathrm{B}(\mathcal{H})$ we describe the relative commutant of $\mathcal{S}$ in the von Neumann algebra generated by $\mathcal{S}$ and G.

1. Introduction

In quantum information theory it turned out to be interesting to know all operators on $\mathrm{B}(\mathcal{H})\overline{\otimes}\mathrm{B}(\mathcal{H})$, where $\mathcal{H}$ is a Hilbert space, that commute with all operators of the form $u \otimes u$ for $u \in \mathrm{B}(\mathcal{H})$ unitary. Specifically, if $\mathcal{H}$ is finite dimensional, it was proved that only such operators are linear combinations of the identity 1 and the flip $V \in \mathrm{B}(\mathcal{H})\overline{\otimes}\mathrm{B}(\mathcal{H})$, where $V \in \mathrm{B}(\mathcal{H} \otimes \mathcal{H}) = \mathrm{B}(\mathcal{H})\overline{\otimes}\mathrm{B}(\mathcal{H})$ is determined by $V(\xi \otimes \eta) = \eta \otimes \xi$ $(\xi, \eta \in \mathcal{H})$ (see [17], [9], [14, Section 7.5]). Since $V^2 = 1 = V^*$, the set $W^*(V) := \mathbb{C}1 + \mathbb{C}V$ is a von Neumann algebra and by the von Neumann bicommutant theorem this result is equivalent to the statement that the commutant of $W^*(V)$ is, as a von Neumann algebra, generated by operators of the form $u \otimes u$, where u is in the unitary group of $\mathrm{B}(\mathcal{H})$. In the next section we give a short proof of this result, valid also in the case when $\mathcal{H}$ is infinite dimensional, in fact we consider the case where $\mathrm{B}(\mathcal{H})$ is replaced by any von Neumann algebra $\mathcal{R}$. Then we generalize this to the case when the flip automorphism of $\mathcal{R}\overline{\otimes}\mathcal{R}$ is replaced by a finite group of automorphisms of a von Neumann algebra. In the special case, when $\mathcal{H}$ is finite-dimensional, $\mathrm{B}(\mathcal{H})\overline{\otimes}\mathrm{B}(\mathcal{H})$ can be identified with the space $\mathrm{L}(\mathrm{B}(\mathcal{H}))$ of all linear maps on $\mathrm{B}(\mathcal{H})$ and it turns out that the above mentioned commutation result can be easily deduced also from a theorem of Brešar [3] concerning commuting mappings. In fact, in Section 3 we will consider a more general situation of

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commuting maps from a von Neumann algebra $\mathcal{R}$ into an $\mathcal{R}$-bimodule. As an application we determine all bounded linear functionals on the operator space projective tensor product $\mathcal{R}\hat{\otimes}\mathcal{R}$ that are invariant under all maps of the form $x \mapsto (u \otimes u)x(u \otimes u)^*$, where $u \in \mathcal{R}$ is unitary, and also all normal such functionals on the von Neumann algebra $\mathcal{R}\overline{\otimes}\mathcal{R}$.

2. The relative commutant of the flip and of a finite unitary group

Let $V \in \mathrm{B}(\mathcal{H} \otimes \mathcal{H})$ be the flip, that is $V(\xi \otimes \eta) = \eta \otimes \xi$ $(\xi, \eta \in \mathcal{H})$, let $\mathcal{F}$ be the automorphism of $\mathrm{B}(\mathcal{H})\overline{\otimes}\mathrm{B}(\mathcal{H}) = \mathrm{B}(\mathcal{H} \otimes \mathcal{H})$ defined by $\mathcal{F}(x) = VxV$ and $\mathcal{F}_{\mathcal{R}}$ its restriction to $\mathcal{R}\overline{\otimes}\mathcal{R}$, where $\mathcal{R} \subseteq \mathrm{B}(\mathcal{H})$ is a von Neumann algebra. Denote $\mathcal{S} = (\mathcal{R}\overline{\otimes}\mathcal{R})'$ so that

$$\mathcal{S}' = \mathcal{R}\overline{\otimes}\mathcal{R},$$

and let $\mathcal{A}_{\mathcal{R}}$ be the von Neumann subalgebra of $\mathcal{S}'$ consisting of all fixed points of $\mathcal{F}_{\mathcal{R}}$. Since $\mathcal{F}_{\mathcal{R}}(a \otimes b) = b \otimes a$ for all $a, b \in \mathcal{R}$, $\mathcal{A}_{\mathcal{R}}$ contains the weak* closure of the set of all finite sums of the form $\sum_i (a_i \otimes b_i + b_i \otimes a_i)$ $(a_i, b_i \in \mathcal{R})$. Conversely, if $w \in \mathcal{S}'$ is fixed by $\mathcal{F}_{\mathcal{R}}$, then $w = \frac{1}{2}(w + \mathcal{F}_{\mathcal{R}}(w))$, hence, approximating w by finite sums of the form $\sum_i a_i \otimes b_i$ $(a_i, b_i \in \mathcal{R})$, we deduce the following simple lemma.

Lemma 2.1. *The fixed point algebra $\mathcal{A}_{\mathcal{R}}$ of $\mathcal{F}_{\mathcal{R}}$ is equal to the weak* closure of the set of all finite sums of the form $\sum_i (a_i \otimes b_i + b_i \otimes a_i)$, where $a_i, b_i \in \mathcal{R}$.*

Denote by $\mathcal{R}_u$ the unitary group of $\mathcal{R}$ and set

$$\mathcal{U}_{\mathcal{R}} := \{u \otimes u : u \in \mathcal{R}_u\}.$$

Lemma 2.2. *$\mathcal{A}_{\mathcal{R}}' = \mathcal{U}_{\mathcal{R}}'$, hence $\mathcal{A}_{\mathcal{R}}$ is generated by $\mathcal{U}_{\mathcal{R}}$ as a von Neumann algebra.*

Proof. By Lemma 2.1 $\mathcal{U}_{\mathcal{R}} \subseteq \mathcal{A}_{\mathcal{R}}$, hence $\mathcal{A}_{\mathcal{R}}' \subseteq \mathcal{U}_{\mathcal{R}}'$, so we need to prove only that each $x \in \mathcal{U}_{\mathcal{R}}'$ is also in $\mathcal{A}_{\mathcal{R}}'$. But x commutes in particular with all elements of the form $e^{ith} \otimes e^{ith}$, where $h \in \mathcal{R}$ is hermitian and $t \in \mathbb{R}$. Using the Taylor expansion $e^{ith} = 1 + ith - \frac{1}{2}t^2h^2 + \ldots$ we see by comparing the linear terms in the identity

$$x(e^{ith} \otimes e^{ith}) = (e^{ith} \otimes e^{ith})x$$

that

$$x(1 \otimes h + h \otimes 1) = (1 \otimes h + h \otimes 1)x.$$

Then x must commute also with all elements in the von Neumann algebra generated by all elements of the form $1 \otimes h + h \otimes 1$, where $h \in \mathcal{R}$ is hermitian, and by Lemma 2.1 this von Neumann algebra is easily seen to be just $\mathcal{A}_{\mathcal{R}}$. □

Observe that by the definitions of $\mathcal{F}_{\mathcal{R}}$ and $\mathcal{A}_{\mathcal{R}}$ we have that $\mathcal{A}_{\mathcal{R}}$ is just the relative commutant of V in $\mathcal{S}'$, that is $\mathcal{A}_{\mathcal{R}} = (V)' \cap \mathcal{S}'$, hence by Lemma 2.2

$$\mathcal{U}_{\mathcal{R}}' = ((V)' \cap \mathcal{S}')' = (V)'' \vee \mathcal{S} \ (= \text{the von Neumann algebra generated by } (V)'' \cup \mathcal{S}). \tag{2.1}$$

By the Tomita commutation theorem ([10], [15]) we have $\mathcal{S} = (\mathcal{R}\overline{\otimes}\mathcal{R})' = \mathcal{R}'\overline{\otimes}\mathcal{R}'$. Further, since $V = V^*$ and $V^2 = 1$, the space $\mathbb{C}1 + \mathbb{C}V$ is a von Neumann algebra and so $(V)'' = \mathbb{C}1 + \mathbb{C}V$. Since $V\mathcal{S}V = \mathcal{S}$, it follows easily that the space $\mathcal{S} + \mathcal{S}V$ is a self-adjoint subalgebra of $\mathrm{B}(\mathcal{H})\overline{\otimes}\mathrm{B}(\mathcal{H})$, containing V and $\mathcal{S}$. It follows from a more general result below (Theorem 2.4) that $\mathcal{S} + \mathcal{S}V$ is weak* closed, hence $\mathcal{S} + \mathcal{S}V \supseteq \mathcal{U}'_{\mathcal{R}}$ by (2.1). Since V and all elements of $\mathcal{S}$ commute with all $u \otimes u$, we also have that $\mathcal{S} + \mathcal{S}V \subseteq \mathcal{U}'_{\mathcal{R}}$, so this proves the following corollary, which in the special case of finite-dimensional situation was proved (by a different method) in [14, p. 105–108] and [17], [9].

Corollary 2.3. *$\{u \otimes u : \ u$ unitary in $\mathcal{R}\}' = \mathcal{S}1 + \mathcal{S}V$, where $\mathcal{S} = \mathcal{R}'\overline{\otimes}\mathcal{R}'$ and V is the flip on $\mathrm{B}(\mathcal{H})\overline{\otimes}\mathrm{B}(\mathcal{H})$ (that is, $V(\xi \otimes \eta = \eta \otimes \xi$, $\xi, \eta \in \mathcal{H}$).*

Let us now consider a more general situation, where $\mathcal{R}\overline{\otimes}\mathcal{R}$ is replaced by a von Neumann algebra $\mathcal{S}$ acting on a Hilbert space $\mathcal{H}$ and the flip V is replaced by a finite group G of unitary operators on $\mathcal{H}$ such that $u\mathcal{S}u^* = \mathcal{S}$ for all $u \in G$.

Theorem 2.4. *If G is finite, the subspace $\mathcal{S}G$ of $\mathrm{B}(\mathcal{H})$ is a von Neumann algebra.*

Proof. Let $\mathcal{A}$ be the implemented crossed product of $\mathcal{S}$ by G (as defined in [10, 13.1.3]). Thus, if n is the cardinality of G and the Hilbert space $\mathcal{H} \otimes \ell^2(G)$ is identified naturally with $\mathcal{H}^n$ (and consequently $\mathrm{B}(\mathcal{H} \otimes \ell^2(G))$ with $\mathrm{M}_n(\mathrm{B}(\mathcal{H}))$), $\mathcal{A}$ consists of all operator matrices of the form $[gh^{-1}s(gh^{-1})]$, where $s : G \to \mathcal{S}$ is any function. (Thus the entry on the position (g, h) is $gh^{-1}s(gh^{-1})$, where $g, h \in G$.) The map

$$\pi : \mathcal{A} \to \mathrm{B}(\mathcal{H}), \quad \pi([gh^{-1}s(gh^{-1})]) = \sum_{t \in G} ts(t)$$

is easily verified to be a $*$-homomorphism, mapping $\mathcal{A}$ onto $G\mathcal{S}$. Further, evidently π is weak* continuous, hence $G\mathcal{S}$ is a von Neumann algebra. □

The commutant of $G\mathcal{S}$ is $G' \cap \mathcal{S}'$, which is just the fixed point algebra of G in $\mathcal{S}'$, that is $(G\mathcal{S})' = \{x \in \mathcal{S}' : gxg^{-1} = x \ \forall g \in G\}$, and coincides also with the range of the projection $x \mapsto \frac{1}{|G|}\sum_{g \in G} gxg^{-1}$ $(x \in \mathcal{S}')$. To determine concretely the relative commutant of $\mathcal{S}$ in $G\mathcal{S}$, we first recall a known decomposition of an automorphism of a von Neumann algebra.

Proposition 2.5. *[10, 12.4.17, 12.4.18] Let $\mathcal{S} \subseteq \mathrm{B}(\mathcal{H})$ be a von Neumann algebra and $v \in \mathrm{B}(\mathcal{H})$ a unitary such that $v\mathcal{S}v^* = \mathcal{S}$. Then there exists a unique central projection $p \in \mathcal{S}$, commuting with v, such that the automorphism $x \mapsto vxv^*$ restricted to $\mathcal{S}p$ is inner, say implemented by a unitary $u \in \mathcal{S}p$, hence $vp = uu'$ for a unitary $u' \in \mathcal{S}'p$, while the action of v on $\mathcal{S}p^\perp$ is free in the sense that $sv \in \mathcal{S}'p^\perp$ $(s \in \mathcal{S})$ implies that $sp^\perp = 0$.*

Now we can describe the relative commutant of $\mathcal{S}$ in $G\mathcal{S}$.

Theorem 2.6. *Let $\mathcal{S} \subseteq \mathrm{B}(\mathcal{H})$ be a von Neumann algebra, $\mathcal{Z}$ the center of $\mathcal{S}$ and G a finite group of unitary operators on $\mathcal{H}$ such that $g\mathcal{S}g^* = \mathcal{S}$ for all $g \in G$. For each $g \in G$ let p_g be the largest projection in $\mathcal{Z}$ such that the automorphism $x \mapsto gxg^*$ on $\mathcal{S}$ is inner, so that*

$$gp_g = u_g u'_g \tag{2.2}$$

for some unitaries $u_g \in p_g\mathcal{S}$ and $u'_g \in p_g\mathcal{S}'$ (see Proposition 2.5). Then

$$G\mathcal{S} \cap \mathcal{S}' = \{\sum_{g \in G} c_g u'_g : c_g \in \mathcal{Z}\}.$$

Proof. Since $\mathcal{Z} \subseteq \mathcal{S}'$ and from (2.2) we have $c_g u'_g = u'_g c_g = g p_g u_g^* c_g \in G\mathcal{S}$, it follows that each element of the form $\sum_{g\in G} c_g u'_g$ (where $c_g \in \mathcal{Z}$) is contained in $G\mathcal{S} \cap \mathcal{S}'$. To prove the reverse inclusion, let $y \in G\mathcal{S} \cap \mathcal{S}'$. As in Theorem 2.4, let $\mathcal{A} \subseteq \mathrm{M}_n(\mathrm{B}(\mathcal{H}))$ be the implemented crossed product of $\mathcal{S}$ by G and let $\pi : \mathcal{A} \to G\mathcal{S}$ be the natural $*$-epimorphism. Since there is a faithful (normal) conditional expectation from $\mathcal{A}$ onto $\mathcal{S}$ of finite index (namely, the map $E : [gh^{-1}s(gh^{-1})] \mapsto s(e)$, which can be easily verified to satisfy the finite index condition $Ex \geq cx\ \forall x\mathcal{A}_+$ for some constant $c > 0$, since n is finite), the inclusion $\mathcal{S} \subseteq \mathcal{A}$ has the relative Dixmier property by [13]. This means that for each $x \in \mathcal{A}$ the closure of the convex hull of the set of all elements of the form uxu^*, where $u \in \mathcal{S}$ is unitary, intersects the commutant of $\mathcal{S}$ in $\mathcal{A}$, hence also the commutant of $\mathcal{S}$ in $\mathrm{M}_n(\mathrm{B}(\mathcal{H}))$, which is $\mathrm{M}_n(\mathcal{S}')$. Choosing $x \in \mathcal{A}$ so that $\pi(x) = y$, it follows that there exists $t \in \mathcal{A} \cap \mathrm{M}_n(\mathcal{S}')$ such that $\pi(t) = y$. Since $t \in \mathcal{A}$, t is of the form $t = [gh^{-1}s(gh^{-1}]$ for a function $s : G \to \mathcal{S}$. Since $t \in \mathrm{M}_n(\mathcal{S}')$, we have that $gh^{-1}s(gh^{-1}) \in \mathcal{S}'$ for all $g, h \in G$, which means just that $gs(g) \in \mathcal{S}'$ for all $g \in G$. With p_g, u_g and u'_g as in the statement of the theorem, we have by Proposition 2.5 that $p_g^\perp s(g) = 0$, hence

$$gs(g) = gp_g s(g) = u'_g u_g s(g).$$

Since $gs(g) \in \mathcal{S}'$, it follows that $u'_g u_g s(g) \in \mathcal{S}'$, hence $u_g s(g) \in u'^*_g \mathcal{S}' \subseteq \mathcal{S}'$. But, since $u_g s(g) \in \mathcal{S}$, this implies that the element $c_g := u_g s(g)$ is in $\mathcal{Z}$. Then $p_g s(g) = p_g u_g^* c_g = u_g^* c_g$. Finally, we compute that

$$y = \pi(t) = \sum_{g\in G} gs(g) = \sum_{g\in G} gp_g s(g) = \sum_{g\in G} u'_g u_g s(g)$$
$$= \sum_{g\in G} u'_g u_g p_g s(g) = \sum_{g\in G} u'_g u_g u_g^* c_g = \sum_{g\in G} u'_g p_g c_g = \sum_{g\in G} c_g u'_g. \quad \square$$

3. Commuting mappings

The tensor product $\mathcal{R}\overline{\otimes}\mathcal{R}$ is dual to the operator space projective tensor product $\mathcal{R}_\sharp\hat{\otimes}\mathcal{R}_\sharp$, where $\mathcal{R}_\sharp$ is the predual of $\mathcal{R}$ ([6, p. 136], [11, p. 49]), and is therefore completely isometrically isomorphic to the space $\mathrm{CB}(\mathcal{R}_\sharp, \mathcal{R})$ of all completely bounded maps from $\mathcal{R}_\sharp$ to $\mathcal{R}$, by the map

$$\iota : \mathrm{CB}(\mathcal{R}_\sharp, \mathcal{R}) \to (\mathcal{R}_\sharp\hat{\otimes}\mathcal{R}_\sharp)^\sharp = \mathcal{R}\overline{\otimes}\mathcal{R}, \quad \iota(\varphi)(\omega\otimes\rho) = (\varphi(\omega))(\rho) \quad (\forall\omega, \rho \in \mathcal{R}_\sharp). \tag{3.1}$$

Under this isomorphism, the condition that an element of $w \in \mathcal{R}\overline{\otimes}\mathcal{R}$ commutes with all elements of the form $u \otimes u$, where $u \in \mathcal{R}$ is unitary, translates into the condition that the corresponding map $\varphi = \iota^{-1}(w)$ satisfies

$$\varphi(u^*\omega u) = u^*\varphi(\omega)u \quad (\forall\omega \in \mathcal{R}_\sharp,\ \forall u \in \mathcal{R} \text{ unitary}).$$

Putting in this identity $u = e^{ith} = 1 + ith + \ldots$, where $h = h^* \in \mathcal{R}$ and $t \in \mathbb{R}$, and comparing the linear terms on both sides, it follows that

$$\varphi([\omega, a]) = [\varphi(\omega), a] \quad (\forall\omega \in \mathcal{R}_\sharp, \forall a \in \mathcal{R}), \tag{3.2}$$

where $[\omega, a]$ denotes the commutator $\omega a - a\omega$. If $\mathcal{R}$ is finite dimensional, then $\mathcal{R}_\sharp$ can be identified with $\mathcal{R}$ and the condition (3.2) simply means that $[\varphi(a), b] = \varphi([a, b])$ for all $a, b \in \mathcal{R}$. In particular

$$[\varphi(a), a] = 0 \quad (\forall a \in \mathcal{R}), \tag{3.3}$$

that is, $\varphi(a)$ commutes with a. Note that replacing in (3.3) a with $a + b$ we get that

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$$[\varphi(a), b] = [a, \varphi(b)] \quad (\forall a, b \in \mathcal{R}). \tag{3.4}$$

Additive mappings satisfying (3.3) on rings were characterized by Brešar [3] and called *commuting mappings* (see also [5, Example 1.5]). In fact, his proof in [3, Lemma 2.2, Corollary 2.3, 1. line on p. 504] works also for mappings from a ring $\mathcal{R}$ into any $\mathcal{R}$-bimodule and shows the following proposition.

Proposition 3.1. *Let $\mathcal{R}$ be a unital ring such that the ideal generated by all commutators $[a, b]$ $(a, b \in \mathbb{R})$ is equal to $\mathcal{R}$ and let $\mathcal{X}$ be an $\mathcal{R}$-bimodule. Let*

$$\mathcal{Z}_{\mathcal{X}} := \{x \in \mathcal{X} : ax = xa \ \forall a \in \mathcal{R}\}$$

be the center of $\mathcal{X}$. Then each additive mapping $\varphi : \mathcal{R} \to \mathcal{X}$ satisfying (3.3) is of the form

$$\varphi(a) = ca + \psi(a), \tag{3.5}$$

where $c \in \mathcal{Z}_{\mathcal{X}} \cap \mathcal{R}\mathcal{X}\mathcal{R}$ and ψ is an additive map from $\mathcal{R}$ to $\mathcal{Z}_{\mathcal{X}}$.

We note that Proposition 3.1 applies in particular to unital C*-algebras which have no tracial states since in such C*-algebras each element is a finite sum of commutators by [12].

In the special case $\mathcal{X} = \mathcal{R}$ it is obvious from (3.4) that φ maps the center of a ring $\mathcal{R}$ into itself. The following example shows that this does not hold any more for mappings into general $\mathcal{R}$-bimodules.

Example 3.2. Let $\mathcal{R}$ be the subalgebra of $\mathrm{M}_3(\mathbb{C})$ (3×3 complex matrices) consisting of all matrices of the form

$$\begin{bmatrix} x & y & z \\ 0 & x & 0 \\ 0 & 0 & x \end{bmatrix}.$$

Define a map $\varphi : \mathcal{R} \to \mathrm{M}_3(\mathbb{C})$ by

$$\varphi\left(\begin{bmatrix} x & y & z \\ 0 & x & 0 \\ 0 & 0 & x \end{bmatrix}\right) = \begin{bmatrix} y & 0 & 0 \\ 0 & y & z \\ 0 & 0 & 0 \end{bmatrix}.$$

It can be verified that $\mathcal{R}$ is abelian, that φ satisfies the condition (3.3), but nevertheless

$$\varphi\left(\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) \text{ does not commute with } \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

A *normal dual bimodule* over a von Neumann algebra $\mathcal{R}$ is a dual Banach space Y such that the module operations $Y \ni y \mapsto ay, ya \in Y$ and $\mathcal{R} \ni a \mapsto ay, ya \in Y$ are weak* continuous (only the weak* continuity of the last two operations will be needed in the proof of the next theorem). In the special case $\mathcal{X} = \mathcal{R}$ the following theorem was proved by Brešar [3]; commuting of maps on C*-algebras were studied by Ara and Mathieu [1], [2].

Theorem 3.3. *Let $\mathcal{X}$ be a subbimodule of a normal dual bimodule over a von Neumann algebra $\mathcal{R}$. Every bounded linear map $\varphi : \mathcal{R} \to \mathcal{X}$, satisfying (3.3), is of the form (3.5), where $\mathcal{Z}_{\mathcal{X}}$ is defined as in Proposition 3.1, $c \in \mathcal{Z}_{\mathcal{X}}$ and ψ is a (bounded linear) map from $\mathcal{R}$ into $\mathcal{Z}_{\mathcal{X}}$.*

Proof. If $\mathcal{R}$ has no abelian central summands, then the theorem is a special case of Proposition 3.1.

Suppose now that $\mathcal{R}$ is abelian. If $\mathcal{R}$ is generated by one element a_0 (which is the case if $\mathcal{R}$ acts on a separable Hilbert space by [15, p. 112]), then it follows from (3.3) (and the weak* continuity of the multiplications $r \mapsto rx, xr$) that $\varphi(a_0)$ commutes with all elements of $\mathcal{R}$, that is $\varphi(a_0) \in \mathcal{Z}_{\mathcal{X}}$. By (3.4) we have $[\varphi(a), a_0] = [a, \varphi(a_0)] = 0 \ \ \forall a \in \mathcal{R}$. Since $\mathcal{R}$ is generated by a_0, this implies that $\varphi(a)$ commutes with all elements of $\mathcal{R}$, that is, $\varphi(a) \in \mathcal{Z}_{\mathcal{X}}$. For a general abelian $\mathcal{R}$, a von Neumann subalgebra $\mathcal{R}_{a,b}$ generated by any two elements a, b is singly generated (it is generated by a countable family of commuting projections, hence an argument from [15, p. 112] shows that it is singly generated). If we apply the above argument to $\mathcal{R}_{a,b}$ instead of $\mathcal{R}$, we see that $[\varphi(a), b] = 0$. Since this holds for all $b \in \mathcal{R}$, this means that $\varphi(a) \in \mathcal{Z}_{\mathcal{X}}$. This proves that $\varphi(\mathcal{R}) \subseteq \mathcal{Z}_{\mathcal{X}}$ if $\mathcal{R}$ is abelian.

In general, let p be the central projection in $\mathcal{R}$ such that $p\mathcal{R}$ is abelian and $p^{\perp}\mathcal{R}$ has no non-zero abelian central summands. Then $p\mathcal{R}$ is contained in the center $\mathcal{Z}$ of $\mathcal{R}$. We claim that $\varphi(\mathcal{Z}) \subseteq \mathcal{Z}_{\mathcal{X}}$. To prove this, let $h = h^* \in \mathcal{R}$. Applying what we have proved in the previous paragraph to the abelian von Neumann algebra $W^*(\mathcal{Z}, h)$ generated by $\mathcal{Z} \cup \{h\}$ and to $\mathcal{X}$ as an $W^*(\mathcal{Z}, h)$-bimodule, we conclude that $[\varphi(z), h] = 0$ for all $z \in \mathcal{Z}$. Thus $[\varphi(z), h + ik] = 0$ for all self-adjoint $h, k \in \mathcal{R}$, meaning that $\varphi(z) \in \mathcal{Z}_{\mathcal{X}}$, that is, $\varphi(\mathcal{Z}) \subseteq \mathcal{Z}_{\mathcal{X}}$. Further, any element z in the center of $p^{\perp}\mathcal{R}$ is also in the center $\mathcal{Z}$ of $\mathcal{R}$, hence $\varphi(z) \in \mathcal{Z}_{\mathcal{X}}$. Since $p^{\perp}\mathcal{R}$ has no non-zero abelian central summands, by Proposition 3.1 there exists

$$c \in \ p^{\perp}\mathcal{X}p^{\perp} \ \text{ satisfying } \ cp^{\perp}a = p^{\perp}ac \ \ \forall a \in \mathcal{R}, \tag{3.6}$$

such that the mapping

$$\psi : \mathcal{R} \to \mathcal{X}, \quad \psi(a) = \varphi(a) - ca$$

satisfies

$$[\psi(p^{\perp}a), p^{\perp}b] = 0 \ \ \forall a, b \in \mathcal{R}. \tag{3.7}$$

Then $c \in \mathcal{Z}_{\mathcal{X}}$ since $[c, a] = [p^{\perp}cp^{\perp}, a] = [p^{\perp}cp^{\perp}, p^{\perp}ap^{\perp}] = 0$ for all $a \in \mathcal{R}$ by (3.6). Further, since $pa \in \mathcal{Z}$ and therefore $\varphi(pa) \in \mathcal{Z}_{\mathcal{X}}$ by what we have already proved, we have

$$[\psi(pa), b] = [\varphi(pa), b] - [cpa, b] = [\varphi(pa), b] = 0 \ \ \forall b \in \mathcal{R}. \tag{3.8}$$

Finally, from (3.8), (3.7), (3.4) and the facts $c \in \mathcal{Z}_{\mathcal{X}}$, $p\mathcal{R} \subseteq \mathcal{Z}$ (hence $\varphi(p\mathcal{R}) \subseteq \mathcal{Z}_{\mathcal{X}}$) we conclude that

$$\begin{aligned}[\psi(a), b] &= [\psi(p^{\perp}a), b] = [\psi(p^{\perp}a), pb] = [\varphi(p^{\perp}a), pb] - [cp^{\perp}a, pb] \\ &= [\varphi(p^{\perp}a), pb] = [p^{\perp}a, \varphi(pb)] = 0.\end{aligned}$$

Thus $\psi(\mathcal{R}) \subseteq \mathcal{Z}_{\mathcal{X}}$. □

Corollary 3.4. *Every bounded linear map $\varphi : \mathcal{R} \to \mathcal{X}$ satisfying*

$$\varphi([a, b]) = [\varphi(a), b] \ \ \forall a, b \in \mathcal{R} \tag{3.9}$$

is of the form (3.5), where $c \in \mathcal{Z}_{\mathcal{X}}$ and $\psi : \mathcal{R} \to \mathcal{Z}_{\mathcal{X}}$ annihilates all commutators $[a, b]$ $(a, b \in \mathcal{R})$. Thus ψ annihilates the properly infinite part of $\mathcal{R}$, while the restriction of ψ to the finite part $\mathcal{R}_f$ of $\mathcal{R}$ is of the form $\psi|\mathcal{R}_f = \rho \circ \tau$, where τ is the central trace on $\mathcal{R}_f$ and ρ is a mapping from the center $\mathcal{Z}_f$ of $\mathcal{R}_f$ into $\mathcal{Z}_{\mathcal{X}}$.

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Proof. By Theorem 3.3 φ is of the form (3.5) and, since $\psi(\mathcal{R}) \subseteq \mathcal{Z}_{\mathcal{X}}$, it follows from (3.9) that

$$\psi([a,b]) = [\psi(a), b] = 0 \ \ \forall a, b \in \mathcal{R}.$$

Since each element in a properly infinite von Neumann algebra is a sum of two commutators by [8], ψ annihilates the properly infinite part of $\mathcal{R}$. On the other hand the restriction of ψ to the finite part $\mathcal{R}_f$ of $\mathcal{R}$ factors as $\psi|\mathcal{R}_f = \tilde{\psi}\eta$, where $\eta : \mathcal{R}_f \to \mathcal{R}_f/[\mathcal{R}_f, \mathcal{R}_f]$ is the quotient map and $\tilde{\psi} : \mathcal{R}_f/[\mathcal{R}_f, \mathcal{R}_f] \to \mathcal{Z}_{\mathcal{X}}$ is the map induced by $\psi|\mathcal{R}_f$; here $[\mathcal{R}_f, \mathcal{R}_f]$ denotes the (closed) subspace generated by the commutators in $\mathcal{R}_f$. By [7] each element with the central trace 0 in a finite von Neumann algebra is a sum of finitely many commutators, hence the central trace τ on $\mathcal{R}_f$ maps $\mathcal{R}_f/[\mathcal{R}_f, \mathcal{R}_f]$ isomorphically onto the center $\mathcal{Z}_f$ of $\mathcal{R}_f$, so that by the open mapping theorem the inverse map $\sigma : \mathcal{Z}_f \to \mathcal{R}_f/[\mathcal{R}_f, \mathcal{R}_f]$ is bounded. Finally observe that $\psi|\mathcal{R}_f = (\tilde{\psi}\sigma)\tau$ and set $\rho = \tilde{\psi}\sigma$. □

After the first version of this paper had already been sent to publication M. Brešar informed me that mappings satisfying (3.9) play a prominent role in the Lie algebra theory. The precise relation between commuting maps and maps satisfying (3.9) is investigated in [4, Theorem 3.1].

Given a bounded linear map φ from a C*-algebra A into a Banach A-bimodule X satisfying (3.4), it can be verified (using the density of A and X in the second duals $A^{\sharp\sharp}$ and $X^{\sharp\sharp}$) that the map $\varphi^{\sharp\sharp} : A^{\sharp\sharp} \to X^{\sharp\sharp}$ also satisfies (3.4). Then by Theorem 3.3 φ must be of the form $\varphi(a) = ca + \psi(a)$, where $c \in \mathcal{Z}_{X^{\sharp\sharp}}$ and ψ is a mapping from A into $\mathcal{Z}_{X^{\sharp\sharp}}$. However, this result is not completely satisfactory since its converse is not true. Perhaps one would like to have c in the center of the multiplier bimodule of X (that is, $c \in \mathrm{M}(X) := \{x \in x^{\sharp\sharp} : xa, ax \in X \ \forall a \in A\}$ and $ca = ac$ for all $a \in A$), but this is not always possible even in the case $X = A$ as shown in [2, Example 6.2.9]. In the case $X = A$ a more precise description of mappings satisfying (3.4) was given by Ara and Mathieu in [1] (see also [2, Section 6.2]) and involves the center of the local multiplier algebra of A. This suggests that one would need local multipliers of Banach bimodules, but, as far as we know, such a theory has not been developed yet. However, if A is abelian, then so is $A^{\sharp\sharp}$, and by the proof of Theorem 3.3 in this case $\varphi^{\sharp\sharp}$ has its range in $\mathcal{Z}_{X^{\sharp\sharp}}$, hence φ must have its range in $\mathcal{Z}_{X^{\sharp\sharp}} \cap X$, which proves the following corollary.

Corollary 3.5. *Every bounded linear commuting mapping φ from an abelian C*-algebra A into a Banach A-bimodule X has its range in the center of X.*

Theorem 3.6. *Let $\mathcal{R}_f$ and $\mathcal{R}_i$ denote the finite and the properly infinite part of a von Neumann algebra $\mathcal{R}$. Every weak* continuous linear functional ω on $\mathcal{R}\overline{\otimes}\mathcal{R}$ which is invariant under all operators of the form $u \otimes u$, where $u \in \mathcal{R}$ is unitary (that is, $\omega((u \otimes u)x(u \otimes u)^*) = \omega(x)$ for all $x \in \mathcal{R}\overline{\otimes}\mathcal{R}$ and unitary $u \in \mathcal{R}$) annihilates $\mathcal{R}_i\overline{\otimes}\mathcal{R} + \mathcal{R}\overline{\otimes}\mathcal{R}_i$, while the restriction $\omega|(\mathcal{R}_f\overline{\otimes}\mathcal{R}_f)$ is of the form*

$$\omega(a \otimes b) = \alpha(\tau_f(a) \otimes \tau_f(b)) + \beta(\tau_f(ab)) \quad (a, b \in \mathcal{R}_f),$$

where τ_f is the central trace on $\mathcal{R}_f$, β is in the predual $(\mathcal{Z}_f)_\sharp$ of the center $\mathcal{Z}_f$ of $\mathcal{R}_f$ and $\alpha \in (\mathcal{Z}_f\overline{\otimes}\mathcal{Z}_f)_\sharp$. Moreover, with $s_\beta \in \mathcal{Z}_f$ the support projection of β, $\mathcal{R}_f s_\beta$ must be a direct sum of finite dimensional factors the dimensions of which are bounded. The converse is also true.

Lemma 3.7. *With the notation as in Theorem 3.6, let $\mathcal{R}\hat{\otimes}\mathcal{R}$ be the operator space projective tensor product and let θ be a bounded linear functional on $\mathcal{R}\hat{\otimes}\mathcal{R}$. Then $\theta((u \otimes u)x(u \otimes u)^*) = \theta(x)$ for all $x \in \mathcal{R}\hat{\otimes}\mathcal{R}$ and all unitary $u \in \mathcal{R}$ if and only if θ annihilates $\mathcal{R}_i \otimes \mathcal{R} + \mathcal{R} \otimes \mathcal{R}_i$ and the restriction $\theta|(\mathcal{R}_f\hat{\otimes}\mathcal{R}_f)$ is of the form*

$$\theta(a \otimes b) = \alpha(\tau_f(a) \otimes \tau_f(b)) + \beta(\tau_f(ab)) \quad (a, b \in \mathcal{R}_f),$$

where β is in the dual $\mathcal{Z}_f^\sharp$ of $\mathcal{Z}_f$ and $\alpha \in (\mathcal{Z}_f \hat{\otimes} \mathcal{Z}_f)^\sharp$.

Proof. Under the natural isomorphism $\iota : \mathrm{CB}(\mathcal{R}, \mathcal{R}^\sharp) \to (\mathcal{R}\hat{\otimes}\mathcal{R})^\sharp$ (which is defined in the same way as (3.1), that is, $\iota(\varphi)(r \otimes s) = (\varphi(r))(s)$, where $r, s \in \mathcal{R}$) functionals $\theta \in (\mathcal{R}\hat{\otimes}\mathcal{R})^\sharp$, that are invariant under all $u \otimes u$ for unitary $u \in \mathcal{R}$, correspond to maps $\varphi \in \mathrm{CB}(\mathcal{R}, \mathcal{R}^\sharp)$, that satisfy $\varphi(uru^*) = u\varphi(r)u^*$ $(r \in \mathcal{R})$, hence satisfy (3.9). (This can be verified by considering u of the form e^{ih}, arguing similarly as in the beginning of this section.) By Corollary 3.4 such a map φ annihilates $\mathcal{R}_i$ (hence $\theta(\mathcal{R}_i \otimes \mathcal{R}) = 0$ and similarly $\theta(\mathcal{R} \otimes \mathcal{R}_i) = 0$), while its restriction to $\mathcal{R}_f$ is of the form $\varphi(a) = \zeta a + \rho \circ \tau_f(a)$, where ζ is in the center $\mathcal{Z}_{\mathcal{R}^\sharp}$ of $\mathcal{R}^\sharp$ and ρ is a linear bounded (hence completely bounded since $\mathcal{Z}_f$ is abelian) map from $\mathcal{Z}_f$ into $\mathcal{Z}_{\mathcal{R}^\sharp}$. Now $\mathcal{Z}_{\mathcal{R}^\sharp}$ consists of all $\sigma \in \mathcal{R}^\sharp$ satisfying $a\sigma = \sigma a$, that is $\sigma(ba) = \sigma(ab)$ for all $a, b \in \mathcal{R}$. Writing σ as $\sigma = \sigma_1 - \sigma_2 + i(\sigma_3 - \sigma_4)$ where all σ_j are positive (in a canonical way, so that σ_1 and σ_2 have orthogonal supports in $\mathcal{R}^{\sharp\sharp}$ and similarly σ_2 and σ_3, see [15, p. 140]) it follows readily that all the σ_j are scalar multiples of tracial states. But there are no such states on the properly infinite part $\mathcal{R}_i$ of $\mathcal{R}$ (since 1 can be written as a sum of two projections both equivalent to 1), hence $\sigma|\mathcal{R}_i = 0$. Thus it follows that $\mathcal{Z}_{\mathcal{R}^\sharp}$ consists of all tracial functionals on $\mathcal{R}_f$, and it is a well-known consequence of the Dixmier approximation theorem that all such functionals are of the form $\beta \circ \tau_f$, where $\beta \in \mathcal{Z}_f^\sharp$. Thus $\mathcal{Z}_{\mathcal{R}^\sharp} = \mathcal{Z}_f^\sharp \circ \tau_f = \mathcal{Z}_{\mathcal{R}_f^\sharp}$ and ζ is of the form $\zeta(a) = \beta(\tau_f(a))$ $(a \in \mathcal{R}_f)$, where $\beta \in \mathcal{Z}_f^\sharp$. For the functional $\theta \in (\mathcal{R}\hat{\otimes}\mathcal{R})^\sharp$ that corresponds to the map $\varphi \in \mathrm{CB}(\mathcal{R}, \mathcal{R}^\sharp)$ under the natural isomorphism $(\mathcal{R}\hat{\otimes}\mathcal{R})^\sharp \cong \mathrm{CB}(\mathcal{R}, \mathcal{R}^\sharp)$ we now have

$$\begin{aligned} \theta(a \otimes b) &= \varphi(a)(b) = (\zeta a)(b) + (\rho(\tau_f(a)))(b) \\ &= \zeta(ab) + (\rho(\tau_f(a)))(b) = \beta(\tau_f(ab)) + (\rho(\tau_f(a)))(b) \end{aligned} \tag{3.10}$$

for all $a, b \in \mathcal{R}_f$, where $\rho(\tau_f(a)) \in \mathcal{Z}_{\mathcal{R}_f^\sharp} = \mathcal{Z}_f^\sharp \circ \tau_f$, hence $\rho(\tau_f(a)) = \gamma(a) \circ \tau_f$, for a functional $\gamma(a) \in \mathcal{Z}_f^\sharp$. Thus from (3.10)

$$\gamma(a)(\tau_f(b)) = (\rho(\tau_f(a)))(b) = \theta(a \otimes b) - \beta(\tau_f(ab)) \quad (a, b \in \mathcal{R}_f). \tag{3.11}$$

Since θ and β are completely bounded maps, it follows readily from (3.11) that $\gamma : \mathcal{R}_f \to \mathcal{Z}_f^\sharp$ is a linear completely bounded map. Further, for each unitary $u \in \mathcal{R}_f$ we have from (3.11) and the invariance of θ that

$$\begin{aligned} \gamma(uau^*)(\tau_f(b)) = \gamma(uau^*)(\tau_f(ubu^*)) &= \theta((u \otimes u)(a \otimes b)(u \otimes u)^* - \beta(\tau_f(uabu^*)) \\ &= \theta(a \otimes b) - \beta(\tau_f(ab)) = \gamma(a)(\tau_f(b)) \quad (a, b \in \mathcal{R}_f), \end{aligned}$$

which implies (since the range of τ_f is $\mathcal{Z}_f$) that $\gamma(uau^*) = \gamma(a)$. Since by the Dixmier approximation theorem the norm closure of the convex hull of the set $\{uau^* : u \text{ unitary in } \mathcal{R}_f\}$ intersects $\mathcal{Z}_f$, and the only point of the intersection is $\tau_f(a)$, it follows that $\gamma(a) = \alpha(\tau_f(a))$, where $\alpha := \gamma|\mathcal{Z}_f \in \mathrm{CB}(\mathcal{Z}_f, \mathcal{Z}_f^\sharp) \cong (\mathcal{Z}_f \hat{\otimes} \mathcal{Z}_f)^\sharp$. From (3.11) we have now (regarding α as an element of $(\mathcal{Z}_f \hat{\otimes} \mathcal{Z}_f)^\sharp$)

$$\theta(a \otimes b) = \alpha(\tau_f(a) \otimes \tau_f(b)) + \beta(\tau_f(ab)) \quad (a, b \in \mathcal{R}_f), \tag{3.12}$$

which proves the lemma in one direction, while the reverse direction is trivial. □

Proof of Theorem 3.6. Let $\omega \in (\mathcal{R}\overline{\otimes}\mathcal{R})^\sharp$ be weak* continuous and invariant (as in the statement of Theorem 3.6). Since the spatial tensor product $\mathcal{R} \otimes \mathcal{R}$ is weak* dense in $\mathcal{R}\overline{\otimes}\mathcal{R}$, ω is determined by the restriction $\omega|(\mathcal{R} \otimes \mathcal{R})$. The natural complete contraction $q : \mathcal{R}\hat{\otimes}\mathcal{R} \to \mathcal{R} \otimes \mathcal{R}$ has dense range, hence ω is determined by the composition $\theta := \omega \circ q$, which has the appropriate form by Lemma 3.7, hence so will ω, if we can show that β is weak* continuous and that α is bounded in the spatial norm of $\mathcal{Z}_f \otimes \mathcal{Z}_f$ and weak* continuous. When $a, b \in \mathcal{Z}_f$, (3.12) (applied to $\omega \circ q$ instead of θ) simplifies to

$$\omega(a \otimes b) = \alpha(a \otimes b) + \beta(ab) \quad (a, b \in \mathcal{Z}_f). \tag{3.13}$$

Since $\mathcal{Z}_f$ is abelian (hence of the form $C(T)$ for a compact space T), the multiplication $\mu : \mathcal{Z}_f \otimes \mathcal{Z}_f \to \mathcal{Z}_f$ is contractive (since μ corresponds to the restriction map $C(T \times T) \ni f \mapsto (f|\Delta) \in C(T)$, where Δ is the diagonal of $T \times T$). Hence $\beta \circ \mu : \mathcal{Z}_f \otimes \mathcal{Z}_f \to \mathbb{C}$ is bounded, and (3.13) implies that α is bounded in the spacial norm, so we can extend α to $\alpha \in (\mathcal{Z}_f \otimes \mathcal{Z}_f)^\sharp$. Then from (3.13) we have $\omega|(\mathcal{Z}_f \otimes \mathcal{Z}_f) = \alpha + \beta \circ \mu$, hence taking the normal parts of maps (see [10, Chapter 10]), we get

$$\omega|(\mathcal{Z}_f \otimes \mathcal{Z}_f) = \alpha_{\text{nor}} + (\beta\mu)_{\text{nor}}.$$

In particular $\beta(z) = \omega(z \otimes 1) - \alpha_{\text{nor}}(z \otimes 1)$ for all $z \in \mathcal{Z}_f$, which implies that β must be weak* continuous. Replacing α with α_{nor} and denoting its weak* continuous extension to $\mathcal{Z}_f \overline{\otimes} \mathcal{Z}_f$ simply by α again, we have now, using (3.12), that $\omega(a \otimes b) = \alpha(\tau_f(a) \otimes \tau_f(b)) + \beta(\tau_f(ab))$ for all $a, b \in \mathcal{R}_f$, however this identity can not, in general, be extended to all elements of $\mathcal{R}_f \otimes \mathcal{R}_f$ since the map $a \otimes b \mapsto \tau_f(ab)$ can not be extended to a bounded map $\mathcal{R}_f \otimes \mathcal{R}_f \to \mathcal{Z}_f$ as will be shown in the next paragraph.

If $e_{i,j} \in \mathcal{R}_f$ $(i, j = 1, \ldots, n)$ are such that $e_{i,j}e_{k,l} = \delta_{j,k}e_{i,l}$, $e_{i,j}^* = e_{j,i}$ and $\sum_{i=1}^n e_{i,i} = 1$, then with $w_n := \sum_{i,j=1}^n e_{i,j} \otimes e_{j,i}$ we have

$$(\tau_f \circ \mu)(w_n) = n\tau_f(\sum_{i=1}^n e_{i,i}) = n,$$

while $w_n = w_n^*$ and $w_n^2 = 1$, hence $\|w_n\| = 1$. Since for each n such an element w can be found in a type II_1 factor, it follows that β must be 0 if $\mathcal{R}_f$ is a type II_1 factor, otherwise $\beta\mu$ would not be bounded. Using the direct integral decomposition one can generalize this to the case when $\mathcal{R}_f$ is not necessarily a facor, but still of type II_1. Alternatively, if $\mathcal{R}_f$ is injective and separable, then by [16, XVI, Corollary 1.43] $\mathcal{R}_f = \mathcal{R}_0 \overline{\otimes} \mathcal{Z}_f$, where $\mathcal{R}_0$ is the injective type II_1 factor, and for a general type II_1 algebra we can consider an injective separable von Neumann subalgebra. If the support s_β of β is not orthogonal to the type II_1 part $\mathcal{R}_2$ of $\mathcal{R}_f$, then $\beta|s_\beta \mathcal{Z}_{\mathcal{R}_2}$ is a nonzero normal functional, hence given by a function $0 \neq g \in L^1(\nu)$ where ν is a positive finite measure on some space such that $s_\beta \mathcal{Z}_{\mathcal{R}_2} \cong L^\infty(\nu)$. With $h \in L^\infty(\nu)$ defined as $h(t) = \overline{g(t)}/|g(t)|$ if $g(t) \neq 0$, and $h(t) = 0$ if $g(t) = 0$, we have

$$\beta\tau_f\mu(h \otimes w_n) = n \int |g(t)|\, d\nu(t) \overset{n\to\infty}{\longrightarrow} \infty,$$

so $\beta\tau_f\mu$ can not be extended to a bounded map on $\mathcal{R}_f \otimes \mathcal{R}_f$ in this case. Thus $s_\beta\mathcal{R}_f$ must be of finite type I, that is, a direct sum of algebras of the form $\mathrm{M}_{n_k}(\mathcal{Z}_k)$, where $\mathcal{Z}_k$ are abelian; moreover essentially the same argument shows that $\sup_k n_k < \infty$. Then $\beta\tau_f\mu$ is bounded, but still not weak* continuous if the centers $\mathcal{Z}_k$ are not atomic. To show this, identify $\mathcal{Z}_k$ with $L^\infty(\nu)$ for a finite positive measure ν on a set Δ. Then $\mathcal{Z}_k \overline{\otimes} \mathcal{Z}_k \cong L^\infty(\nu \times \nu)$, $\beta|\mathcal{Z}_k$ is given by a function $g \in L^1(\nu)$ and the map $\beta\tau_f\mu|(\mathcal{Z}_k \overline{\otimes} \mathcal{Z}_k)$ is given by $h \mapsto \int h(t,t)g(t)\, d\nu(t)$. If ν has no atoms, then by considering a sequence of suitable functions h_n the supports of which are concentrated nearer and nearer the diagonal of $\Delta \times \Delta$, we see that $\beta\tau_f\mu$ can not be weak* continuous. Thus, if $\beta\tau_f\mu$ is weak* continuous, the non-atomic part of ν must be absent, hence ν must be atomic. This proves the theorem in one direction. The reverse direction follows from the weak* continuity of the central traces τ_f and $\tau_f \overline{\otimes} \tau_f$ and the weak* continuity of multiplication on atomic abelian von Neumann algebras. (The multiplication $\ell^\infty \overline{\otimes} \ell^\infty \to \ell^\infty$ is the second adjoint to the multiplication $c_0 \otimes c_0 \to c_0$, hence weak* continuous.) □

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